From Laplacian Transport to Dirichlet-to-Neumann (Gibbs) Semigroups

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FROM LAPLACIAN TRANSPORT TO DIRICHLET-TO-NEUMANN (GIBBS) SEMIGROUPS

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Abstract[3]

The paper gives a short account of some basic properties of *Dirichlet-to-Neumann* operators $\Lambda_{\gamma,\partial\Omega}$ including the corresponding semigroups motivated by the Laplacian transport in anisotropic media ($\gamma \neq I$) and by elliptic systems with dynamical boundary conditions. For illustration of these notions and the properties we use the explicitly constructed *Lax semigroups*. We demonstrate that for a general smooth bounded convex domain $\Omega \subset \mathbb{R}^d$ the corresponding Dirichlet-to-Neumann semigroup $\left\{U(t) := e^{-t\Lambda_{\gamma,\partial\Omega}}\right\}_{t\geq 0}$ in the Hilbert space $L^2(\partial\Omega)$ belongs to the *trace-norm* von Neumann-Schatten ideal for any $t > 0$. This means that it is in fact an *immediate Gibbs* semigroup. Recently Emamirad and Laadnani have constructed a *Trotter-Kato-Chernoff* product-type approximating family $\{(V_{\gamma,\partial\Omega}(t/n))^n\}_{n\geq 1}$ *strongly* converging to the semigroup $U(t)$ for $n \to \infty$. We conclude the paper by discussion of a conjecture about convergence of the *Emamirad-Laadnani approximantes* in the the *trace-norm* topology.

Key words: Laplacian transport, Dirichlet-to-Neumann operators, Lax semigroups, Dirichlet-to-Neumann semigroups, Gibbs semigroups.
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Contents

1. Laplacian transport and Dirichlet-to-Neumann operators

Example 1.1. Is is well-known (see e.g. [LeUl]) that the problem of determining a *conductivity matrix* field $\gamma(x) = [\gamma_{i,j}(x)]_{i,j=1}^{d}$, for x in a bounded open domain $\Omega \subset \mathbb{R}^d$, is related to "measuring" the elliptic *Dirichlet-to-Neumann* map for associated conductivity equation. Notice that solution of this problem has a lot of practical applications in various domains: geophysics, electrochemistry etc. It is also an important diagnostic tool in medicine, e.g. in the *electrical impedance tomography*; the tissue in the human body is an example of highly anisotropic conductor [BaBr].

Under assumption that there is no sources or sinks of current the potential $v(x)$, $x \in \Omega$, for a given voltage $f(\omega)$, $\omega \in \partial\Omega$, on the (smooth) boundary $\partial\Omega$ of Ω is a solution of the Dirichlet problem:

$$(\mathbf{P1}) \qquad \begin{cases} \operatorname{div}(\gamma \nabla v) = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases}$$

Then the corresponding to ($\mathbf{P1}$) Dirichlet-to-Neumann map (operator) $\Lambda_{\gamma,\partial\Omega}$ is defined by

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f / \partial \nu_\gamma := \nu \cdot \gamma \ \nabla v_f \ |_{\partial\Omega} \ . \tag{1.1}$$

Here ν is the unit outer-normal vector to the boundary at $\omega \in \partial\Omega$ and the function $u := u_f$ is solution of the Dirichlet problem ($\mathbf{P1}$).

The Dirichlet-to-Neumann operator (1.1) is also called the *voltage-to-current* map, since the function $\Lambda_{\gamma,\partial\Omega} f$ gives the induced current flux trough the boundary $\partial\Omega$. The key (*inverse*) problem is whether on can determine the conductivity matrix γ by knowing electrical boundary measurements, i.e. the corresponding Dirichlet-to-Neumann operator? Unfortunately, this operator does not determine the matrix γ uniquely, see e.g. [GrUl] and references there.

Example 1.2. The problem of electrical current flux in the form ($\mathbf{P1}$) is an example of so-called *Laplacian transport*. Besides the voltage-to-current problem the motivation to study this kind of transport comes for instance from the *transfer* across biological membranes, see e.g. [Sap], [GrFiSap].

Let some "species" of concentration $C(x)$, $x \in \mathbb{R}^d$, diffuse in the *isotropic* bulk ($\gamma = I$) from a (distant) source localised on the closed boundary $\partial\Omega_0$ towards a *semipermeable* compact interface $\partial\Omega$ on which they disappear at a given rate W. Then the *steady* concentration field (Laplacian transport with a diffusion coefficient D) obeys the set of equations:

$$(\mathbf{P2})\qquad \begin{cases} \Delta C = 0,\ x \in \Omega_0 \setminus \Omega\ , \\ C(\omega_0 \in \partial\Omega_0) = C_0,\ \text{at the source}\ , \\ D\ \partial_\nu C(\omega) = W\ (C(\omega) - 0),\ \text{on the interface } \omega \in \partial\Omega \end{cases}$$

Let $C = C_0(1-u)$. Then $\Delta u = 0$, $x \in \Omega$. If we put $\mu := D/W$, then the boundary conditions on $\partial\Omega$ take the form: $(I - \mu\partial_\nu)u\ |_{\partial\Omega}\ (\omega) = 1\ |_{\partial\Omega}\ (\omega)$, where $(1\ |_{\partial\Omega})(\omega) = \chi_{\partial\Omega}(\omega)$ is characteristic function of the set $\partial\Omega$, and $u(\omega_0) = 0, \omega_0 \in \partial\Omega_0$ on the source boundary.

Consider now the following auxiliary Laplace-Dirichlet problem:

$$\Delta u = 0,\ x \in \Omega_0 \setminus \Omega\ ,\quad u\ |_{\partial\Omega}\ (\omega) = f(\omega \in \partial\Omega)\ \text{ and }\ u\ |_{\partial\Omega_0}\ (\omega) = 0\ , \tag{1.2}$$

with solution u_f. Then similar to (1.1) we can associate with the problem (1.2) a Dirihlet-to-Neumann operator

$$\Lambda_{\gamma=I,\partial\Omega} : f \mapsto \partial_\nu u_f\ |_{\partial\Omega} \tag{1.3}$$

with domain $\mathrm{dom}(\Lambda_{I,\partial\Omega})$, which belongs to a certain *Sobolev* space, Section 2.

The advantage of this approach is that as soon as the operator (1.3) is defined one can apply it to study the *mixed* boundary value problem (**P2**). This gives in particular the value of the particle flux due to Laplacian transport across the membrane $\partial\Omega$. Indeed, one obtain that $(I + \mu\Lambda_{I,\partial\Omega})u\ |_{\partial\Omega} = 1\ |_{\partial\Omega}$, and that the local (diffusive) particle *flux* is defined as:

$$\phi\ |_{\partial\Omega} := D\ C_0(-\partial_n u)\ |_{\partial\Omega} = D\ C_0(\Lambda_{I,\partial\Omega}(I + \mu\Lambda_{I,\partial\Omega})^{-1}1)\ |_{\partial\Omega}\ . \tag{1.4}$$

Then the corresponding total flux across the membrane $\partial\Omega$:

$$\Phi := (\phi, 1)_{L^2(\partial\Omega)} = D\ C_0(\Lambda(I + \mu\Lambda_{I,\partial\Omega})^{-1}1, 1)_{L^2(\partial\Omega)} \tag{1.5}$$

is experimentally measurable macroscopic response of the system, expressed via transport parameters D, C_0, μ and geometry of $\partial\Omega$. Here $(\cdot,\cdot)_{L^2(\partial\Omega)}$ is scalar product in the Hilbert space $\partial\mathcal{H} := L^2(\partial\Omega)$.

The aim of the present paper is twofold:
(i) to give a short account of some standard results about Dirichlet-to-Neumann operators and related *Dirichlet-to-Neumann semigroups* that solve a certain class of elliptic systems with dynamical boundary conditions;
(ii) to present some recent results concerning the *approximation* theory and the *Gibbs* character of the Dirichlet-to-Neumann semigroups for compact sets Ω with smooth boundaries $\partial\Omega$.

To this end in the next Section 2 we recall some fundamental properties of the Dirichlet-to-Neumann operators and semigroups, we illustrate them by few elementary examples, including the *Lax semigroups* [Lax].

In Section 3 we present the strong *Emamirad-Laadnani approximations* of the Dirichlet-to-Neumann semigroups inspired by the *Chernoff* theory and by its generalizations in [NeZag], [CaZag2].

We show in Section 4 that for compact sets Ω with smooth boundaries $\partial\Omega$ the Dirichlet-to-Neumann semigroups are in fact (immediate) *Gibbs* semigroups [Zag2].

Some recent results and conjectures about approximations of the Dirichlet-to-Neumann (Gibbs) semigroups in operator and *trace-norm* topologies are collected in the last Section 5.

2. Dirichlet-to-Neumann operators and semigroups

2.1 Dirichlet-to-Neumann operators

Let Ω be an open bounded domain in $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$. Let γ be a $C^\infty(\overline{\Omega})$ matrix-valued function on $\overline{\Omega}$, which we call the *Laplacian transport matrix* in domain Ω.

We suppose that the matrix-valued function $\gamma(x) := [\gamma_{i,j}(x)]_{i,j=1}^d$ satisfies the following hypotheses:
(H1) The real coefficients are symmetric and $\gamma_{i,j}(x) = \gamma_{j,i}(x) \in \mathcal{C}^\infty(\overline{\Omega})$.
(H2) There exist two constants $0 < c_1 \leq c_2 < \infty$ such that for all $\xi \in \mathbb{R}^d$, we have

$$c_1\|\xi\|^2 \leq \sum_{i,j=1}^{n} \xi_i\xi_j\gamma_{i,j}(x) \leq c_2\|\xi\|^2. \tag{2.1}$$

Then the *Dirichlet-to-Neumann* operator $\Lambda_{\gamma,\partial\Omega}$ associated with the Laplacian transport in Ω is defined as follows.

Let $f \in C(\partial\Omega)$ and denote by v_f the *unique* solution (see e.g. [GiTr], Theorem 6.25) of the Dirichlet problem

$$(\mathbf{P1}) \qquad \begin{cases} A_{\gamma,\partial\Omega}\ v := \operatorname{div}(\gamma\ \nabla v) = 0 \quad \text{in } \ \Omega\ , \\ v\ |_{\partial\Omega}= f \quad \text{on } \ \partial\Omega\ , \end{cases}$$

in the Banach space $X := C(\overline{\Omega})$. Here operator $A_{\gamma,\partial\Omega}$ is defined on its maximal domain

$$\operatorname{dom}(A_{\gamma,\partial\Omega}) := \{u \in X : A_{\gamma,\partial\Omega}\ u \in X\}\ . \tag{2.2}$$

Definition 2.1. *The Dirichlet-to-Neumann operator is the map*:

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f/\partial\nu_\gamma = \nu\cdot\gamma\nabla v_f\ |_{\partial\Omega}\ , \tag{2.3}$$

with domain :

$$\begin{aligned} &\operatorname{dom}(\Lambda_{\gamma,\partial\Omega}) = \\ &\{f \in \partial C(\Omega_R) : v_f \in \operatorname{Ker}(A_{\gamma,\partial\Omega}) \ \text{ and } \ |(\nu\cdot\gamma\nabla v_f\ |_{\partial\Omega})| < \infty\}. \end{aligned} \tag{2.4}$$

Here ν denotes the unit outer-normal vector at $\omega \in \partial\Omega$ and v_f is the solution of Dirichlet problem **(P1)**.

The solution $v_f := L_{\partial\Omega} f$ of the problem **(P1)** is called the γ-*harmonic lifting* of f, where $L_{\partial\Omega} : C(\partial\Omega) \mapsto C^2(\Omega) \cap C(\overline{\Omega})$ is called the *lifting* operator with domain $\mathrm{dom}(L_{\partial\Omega}) = C(\partial\Omega)$. If $T_{\partial\Omega} : C(\overline{\Omega}) \mapsto C(\partial\Omega)$ denotes the *trace* operator on the smooth boundary $\partial\Omega$, i.e. $v\,|_{\partial\Omega} = T_{\partial\Omega}\, v$, then [Eng]:

$$L_{\partial\Omega} = (T_{\partial\Omega}\,|_{\mathrm{Ker}(A_{\gamma,\partial\Omega})})^{-1} \;\text{ and }\; \mathrm{dom}(\Lambda_{\gamma,\partial\Omega}) = T_{\partial\Omega}\{\mathrm{Ker}(A_{\gamma,\partial\Omega})\}\ . \tag{2.5}$$

Remark 2.2. *Let $\partial X := C(\partial\Omega)$. Then (2.5) implies :*

$$T_{\partial\Omega}L_{\partial\Omega}\; u = u\ ,\ \ u \in \partial X \;\text{ and }\; L_{\partial\Omega}T_{\partial\Omega}\; w = w\ ,\ \ w \in \mathrm{Ker}(A_{\gamma,\partial\Omega})\ . \tag{2.6}$$

One also gets that the lifting operator is bounded: $L_{\partial\Omega} \in \mathcal{L}(\partial X, X)$, whereas the Dirichlet-to-Neumann operator (2.3) is obviously not.

Now let $\mathcal{H}$ be Hilbert space $L^2(\Omega)$ and $\partial\mathcal{H} := L^2(\partial\Omega)$ denote the *boundary space*. In order that the problem **(P1)** admits a *unique* solution v_f, one has to assume that $f \in W_2^{1/2}(\partial\Omega)$, and then v_f belongs the *Sobolev* space $W_2^1(\Omega)$, see e.g. [Tay, Ch.7]. So, we can define Dirichlet-to-Neumann operator in the Hilbert space $\partial\mathcal{H}$ by (2.3) with domain:

$$\mathrm{dom}(\Lambda_{\gamma,\partial\Omega}) := \{f \in W_2^{1/2}(\partial\Omega) :\ \Lambda_{\gamma,\partial\Omega} f \in \partial\mathcal{H} = L^2(\partial\Omega)\}. \tag{2.7}$$

Proposition 2.3. *The Dirichlet-to-Neumann operator (2.3) with domain (2.7) in the Hilbert space $\partial\mathcal{H}$ is unbounded, non-negative, self-adjoint, first-order elliptic pseudo-differential operator with compact resolvent.*

The complete proof can be found e.g. in [Tay, Ch.7], [Tay1]. Therefore, we give here only some comments on these properties of the Dirichlet-to-Neumann operator (2.3) in $\partial\mathcal{H} = L^2(\partial\Omega)$.

Remark 2.4. (a) *By virtue of definition (2.3) for any $f \in W_2^{1/2}(\partial\Omega)$ one gets:*

$$(f, \Lambda_{\gamma,\partial\Omega} f)_{\partial\mathcal{H}} = \int_{\partial\Omega} d\sigma(\omega)\ \overline{v_f(\omega)}\ \nu\cdot\gamma(\omega)(\nabla v_f)(\omega) = \tag{2.8}$$
$$\int_{\Omega} dx\ \ \mathrm{div}(\overline{v_f(x)}\ (\gamma\nabla v_f)(x)) = \int_{\Omega} dx\ (\nabla\overline{v_f(x)}\cdot\gamma\ \nabla v_f)(x)) \geq 0\ ,$$

since the matrix γ verifies **(H2)***. Thus, operator $\Lambda_{\gamma,\partial\Omega}$ is non-negative.*
(b) *In fact to ensure the existence of the trace $T_{\partial\Omega}(\nu\cdot\gamma\nabla(L_{\partial\Omega} f))$ one has initially to define operator $\Lambda_{\gamma,\partial\Omega}$ for $f \in W_2^{3/2}(\partial\Omega)$. Then Dirichlet-to-Neumann operator is a self-adjoint extension with domain (2.7) and moreover it is a bounded map $\Lambda_{\gamma,\partial\Omega} : W_2^{1/2}(\partial\Omega) \mapsto W_2^{-1/2}(\partial\Omega)$.*
(c) *By (2.8) and since derivatives of the first-order are involved in (2.3) one can conclude that this operator should be elliptic and pseudo-differential. If $\gamma(x) = I$, then $\Lambda_{I,\partial\Omega}$ is, roughly, the operator $(-\Delta_{\partial\Omega})^{1/2}$, where $\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$, with corresponding induced metric* [Tay, Ch.7], [Tay1].
(d) *Compactness of the imbedding $W_2^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ implies the compactness of the resolvent of $\Lambda_{\gamma,\partial\Omega}$.*

By (a) and (d) the spectrum $\sigma(\Lambda_{\gamma,\partial\Omega})$ of the Dirichlet-to-Neumann operator is a set of non-negative increasing eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. The rate of increasing is given by the Weyl asymptotic formula, see e.g. [Hor], [Tay]:

Proposition 2.5. *Let $\Lambda_{\gamma,\partial\Omega}(x,\xi)$, for $(x,\xi) \in T^*\partial\Omega$, be the symbol of the first-order, elliptic pseudo-differential Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$. Then the asymptotic behaviour of the corresponding eigenvalues as $k \to \infty$ has the form:*

$$\lambda_k \sim \left\{ \frac{k}{C(\partial\Omega, \Lambda_\gamma)} \right\}^{1/(d-1)} ,$$

where

$$C(\partial\Omega, \Lambda_\gamma) := \frac{1}{(2\pi)^{d-1}} \int_{\Lambda_{\gamma,\partial\Omega}(x,\xi)\leq 1} dx\, d\xi \ .$$

Another important result is due to Hislop and Lutzer [HiLu]. It concerns a localization (*rapid decay*) of the γ-harmonic lifting of the corresponding eigenfunctions.

Proposition 2.6. *Let $\{\phi_k\}_{k=1}^{\infty}$ be eigenfunctions of the Dirichlet-to-Neumann operator: $\Lambda_{\gamma,\partial\Omega} = \lambda_k \phi_k$ with $\|\phi_k\|_{L^2(\partial\Omega)} = 1$. Let $v_{\phi_k} := L_{\partial\Omega}\phi_k$ be γ-harmonic lifting of ϕ_k to Ω corresponding to the problem* **(P1)**. *Then for any compact $\mathcal{C} \subset \phi_k$ and $x \in \mathcal{C}$ one gets the representation:*

$$|v_{\phi_k}(x)| = \psi(x,p,\mathcal{C})/{\lambda_k}^p \tag{2.9}$$

with arbitrary large $p > 0$. Here $\psi(x,p,\mathcal{C})$ is a decreasing function of the distance $\mathrm{dist}(x,\, \partial\Omega)$.

Since by the Weyl asymptotic formula we have $\lambda_k = O(k^{1/(d-1)})$, the decay implied by the estimate (2.9) is algebraic.

Conjecture 2.7. [HiLu] *In fact the order of decay instead of $\psi(x,p,\mathcal{C})/{\lambda_k}^p$ is exponential: $O(\exp[-\,k\,\mathrm{dist}(\mathcal{C},\, \partial\Omega)])$.*

2.2 Example of a Dirichlet-to-Neumann operator

To illustrate the results mentioned above we consider a simple example which will be useful below for contraction of the *Lax semigroups*.

Consider a homogeneous isotropic case: $\gamma(x) = I$, and let $\Omega = \Omega_R := \{x \in \mathbb{R}^{d=3} :\ \|x\| < R\}$. Then $A_{\gamma,\partial\Omega_R} = \Delta_{\partial\Omega_R}$ and for the *harmonic lifting* of

$$f(\omega) = \sum_{l,m} f_{l,m}^{(R)}\ Y_{l,m}(\theta,\varphi) \in W_2^{1/2}(\partial\Omega_R) \ ,$$

we obtain:

$$v_f(r,\theta,\varphi) = \sum_{l,m} \left(\frac{r}{R}\right)^l\ f_{l,m}^{(R)}\ Y_{l,m}(\theta,\varphi), \tag{2.10}$$

since the *spherical* functions $\{Y_{l,m}\}_{l=0,|m|\leq l}^{\infty}$ form a complete orthonormal basis in the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R,\ d\theta\, \sin\theta\, d\varphi)$.

Definition (2.3) and (2.10) imply that non-negative, self-adjoint, first-order elliptic pseudo-differential Dirichlet-to-Neumann operator

$$(\Lambda_{I,\partial\Omega_R} f)(\omega = (R,\theta,\varphi)) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\left(\frac{l}{R}\right)\ f_{l,m}^{(R)}\ Y_{l,m}(\theta,\varphi), \tag{2.11}$$

has discrete spectrum $\sigma(\Lambda_{I,\partial\Omega_R}) := \{\lambda_{l,m} = \ l/R\}_{l=0,|m|\leq l}^{\infty}$ with spherical eigenfunctions:

$$(\Lambda_{I,\partial\Omega_R} Y_{l,m})(R,\theta,\varphi) = \left(\frac{l}{R}\right)\ Y_{l,m}(\theta,\varphi), \tag{2.12}$$

and multiplicity m. The operator (2.11) is obviously unbounded and it has a compact resolvent.

Remark 2.8. *Since by virtue of (2.10) the γ-harmonic lifting of the eigenfunction $Y_{l,m}$ to the ball Ω_R is*

$$v_{Y_{l,m}}(r,\theta,\varphi) = \left(\frac{r}{R}\right)^l\ Y_{l,m}(\theta,\varphi)\ ,$$

one can check the localization (Proposition 2.6) and Conjecture about the exponential decay explicitly. For distances: $0 < \operatorname{dist}(x,\partial\Omega_R) = R - r \ll R$*, one obtains* $|v_{Y_{l,m}}(r,\theta,\varphi)| = O(e^{-l(R-r)/R})$*.*

2.3 Dirichlet-to-Neumann semigroups on ∂X

To define the Dirichlet-to-Neumann semigroups on the *boundary* Banach space $\partial X = C(\partial\Omega)$ we can follow the line of reasoning of [Esc] or [Eng]. To this end consider in $X = C(\Omega)$ the following elliptic system with *dynamical boundary conditions*

$$(\mathbf{P2})\qquad \begin{cases} \operatorname{div}(\gamma\nabla u(t,\cdot)) = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial u(t,\cdot)/\partial t + \partial u(t,\cdot)/\partial\nu_\gamma = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega. \end{cases}$$

Proposition 2.9. *The problem* **(P2)** *has a unique solution $u_f(t,x)$ for any $f \in C(\partial\Omega)$. Its trace on the boundary $\partial\Omega$ has the form:*

$$u_f(t,\omega) := (T_{\partial\Omega}u_f(t,\cdot))(\omega) = (U(t)f)(\omega)\ , \tag{2.13}$$

where the family of operators $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ is a C_0-semigroup generated by the Dirichlet-to-Neumann operator of the problem **(P1)***.*

The following key result about the properties of the Dirichlet-to-Neumann semigroups on the boundary Banach space $\partial X = C(\partial\Omega)$ is due to Escher-Engel [Esc],[Eng] and Emamirad-Laadnani [EmLa]:

Proposition 2.10. *The semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ is analytic, compact, positive, irreducible and Markov C_0-semigroup of contractions on $C(\partial\Omega)$.*

Remark 2.11. *The complete proof can be found in the papers quoted above. So, here we make only some comments and hints concerning the Proposition 2.10.*

2.4 Dirichlet-to-Neumann semigroups on $\partial\mathcal{H}$

The Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ on $\partial\mathcal{H}$ is defined by self-adjoint and non-negative Dirichlet-to-Neumann generator $\Lambda_{\gamma,\partial\Omega}$ of Proposition 2.3.

Proposition 2.12. *The Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_t$ on the Hilbert space $\partial\mathcal{H}$ is a holomorphic quasi-sectorial contraction with values in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$ for $\mathrm{Re}\,(t) > 0$.*

Remark 2.13. *The first part of the statement follows from Proposition 2.3. Since the generator $\Lambda_{\gamma,\partial\Omega}$ is self-adjoint and non-negative, the semigroup $\{U(t)\}_t$ is holomorphic and quasi-sectorial contraction for $\mathrm{Re}\,(t) > 0$, see e.g.* [CaZag1], [Zag1]*. Compactness of the resolvent of $\Lambda_{\gamma,\partial\Omega}$ implies the compactness of $\{U(t)\}_{t>0}$, but to prove the last part of the statement we need a supplementary argument about asymptotic behaviour of its eigenvalues given by the Weyl asymptotic formula, Proposition 2.5.*

This behaviour of eigenvalues implies the second part of the Proposition 2.12:

Lemma 2.14. *The Dirichlet-to-Neumann semigroup $U(t)$ has values in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$ for any $t > 0$.*

Proof. Since the Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$ is self-adjoint, we have to prove that

$$\|U(t)\|_1 = \sum_{k\geq 1} \mathrm{e}^{-t\lambda_k} < \infty \ , \tag{2.14}$$

for $t > 0$. Here $\|\cdot\|_1$ denotes the norm in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$. Then the Weyl asymptotic formula implies that there exists bounded M and function $r(k)$ such that

$$\begin{aligned}\sum_{k\geq 1} \mathrm{e}^{-t\lambda_k} &\leq \sum_{k\geq 1} \exp\{-t[(k/c)^{\frac{1}{d-1}} + r(k)]\} \\ &\leq \mathrm{e}^{tM} \sum_{k\geq 1} \exp\{-t(k/c)^{\frac{1}{d-1}}\}.\end{aligned}$$

Here $c := C(\partial\Omega, \Lambda_\gamma)$ and the last sum converges for any $t > 0$, which proves the equation (2.14). □

2.5 Example: Lax semigroups

A beautiful example of explicit representation of the Dirichlet-to-Neumann semigroup (2.13) is due to Lax [Lax], Ch.36.

Let $\gamma(x) = I$, and $\Omega = \Omega_R$, see Section 2.2. Following [Lax] we define the mapping:

$$K(t) : v(x) \mapsto v(e^{-t/R}\, x) \ \text{ for any } \ u \in C(\Omega_R) \ , \tag{2.15}$$

which is a semigroup for the parameter $t \geq 0$ in the Banach space $X = C(\Omega_R)$:

$$(K(\tau)K(t)v)(x) = v(e^{-\tau/R}\, e^{-t/R}\, x) = v(e^{-(\tau+t)/R}\, x) \ , \ \tau, t \geq 0 \ , \ x \in \Omega_R \ . \tag{2.16}$$

Remark 2.15. *It is clear that if $v(x)$ is $(\gamma = I)$-harmonic in $C(\Omega_R)$, then the function: $x \mapsto v(e^{-t/R}\, x)$ is also harmonic. Therefore,*

$$u_f(t,x) := v_f(e^{-t/R}\, x) = (K(t)L_{\partial\Omega_R} f)(x) = (L_{\partial\Omega_R} f_t)(x) \ , \ x \in \Omega_R \ , \qquad (2.17)$$

is the harmonic lifting of the function $f_t(\omega) := v_f(e^{-t/R}\, \omega)$, $\omega \in \partial\Omega_R$, where v_f solves the problem **(P1)** *for $\gamma = I$. Since in the spherical coordinates $x = (r, \theta, \varphi)$ one has:*

$$\partial v_f(t,x)/\partial t = -\partial_r v_f(e^{-t/R} r, \theta, \varphi) e^{-t/R}\ (r/R)$$

and

$$\partial v_f(t, R, \theta, \varphi)/\partial \nu_I = \partial_r v_f(e^{-t/R} r, \theta, \varphi) e^{-t/R} \ ,$$

we get that $\partial u_f(t,\omega)/\partial t + \partial u_f(t,\omega)/\partial \nu_I = 0$, i.e. the function (2.17) is a solution of the problem **(P2)***.*

Hence, according to (2.13) and (2.17) the operator family:

$$S(t) := T_{\partial\Omega_R} K(t) L_{\partial\Omega_R} \ , \ t \geq 0 \ , \qquad (2.18)$$

defines the Dirichlet-to-Neumann semigroup corresponding to the problem **(P2)** for $\gamma(x) = I$, and $\Omega = \Omega_R$, which is known as the *Lax semigroup*. By virtue of (2.17) and (2.18) the action of this semigroup is known explicitly:

$$(S(t)f)(\omega) = v_f(e^{-t/R}\omega) \ , \ \omega \in \partial\Omega_R \ . \qquad (2.19)$$

Notice that the semigroup relation:

$$S(\tau)S(t) = T_{\partial\Omega_R} K(\tau) L_{\partial\Omega_R} T_{\partial\Omega_R} K(t) L_{\partial\Omega_R} = S(\tau + t) \ , \qquad (2.20)$$

follows from the properties of lifting and trace operators (see Remark 2.2), from identity (2.16) and definition (2.18). One finds generator $\Lambda_{\gamma=I,\partial\Omega_R}$ of this semigroup from the limit:

$$0 = \lim_{t\to 0} \sup_{\omega\in\partial\Omega_R} |\frac{1}{t}(f - S(t)f)(\omega) - (\Lambda_{\gamma=I,\partial\Omega_R} f)(\omega)| = \qquad (2.21)$$

$$\lim_{t\to 0} \sup_{\omega\in\partial\Omega_R} |\frac{1}{t}(v_f(R,\theta,\varphi) - v_f(e^{-t/R}\, R,\theta,\varphi)) - (\Lambda_{\gamma=I,\partial\Omega_R} f)(R,\theta,\varphi)|.$$

Then operator

$$(\Lambda_{\gamma=I,\partial\Omega_R} f)(R,\theta,\varphi) = \partial_r v_f(r = R, \theta, \varphi) \qquad (2.22)$$

for any function f from domain:

$$\mathrm{dom}(\Lambda_{I,\partial\Omega_R}) = \{f \in \partial C(\Omega_R) : v_f \in \mathrm{Ker}(A_{I,\partial\Omega_R}) \,\mathrm{and}\, |(\partial_r v_f)\, |_{\partial\Omega_R}\, | < \infty\} \quad (2.23)$$

is identical to (2.4) for the case: $\gamma = I$ and $\partial\Omega = \partial\Omega_R$. Therefore, generator (2.22) of the Lax semigroup is the the Dirichlet-to-Neumann operator in this particular case of the Banach space $\partial X = C(\partial\Omega_R)$.

Similarly we can consider the Lax semigroup (2.18) in the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R,\ d\theta\, \sin\theta\, d\varphi)$. Since generator of this semigroup is a particular case of the Dirichlet-to-Neumann operator (2.11), by (2.12) and (2.10) we again obtain the corresponding action in the explicit form:

$$
\begin{aligned}
&(S(t)f)(\omega) = \qquad (2.24)\\
&(e^{-t\Lambda_{I,\partial\Omega_R}} f)(\omega)) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\sum_{s=0}^{\infty} \frac{(-t)^s}{s!}\left(\frac{l}{R}\right)^s \; f^{(R)}_{l,m}\; Y_{l,m}(\theta,\varphi) = \\
&\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} (e^{-\,t/R})^l \; f^{(R)}_{l,m}\; Y_{l,m}(\theta,\varphi) = v_f(e^{-t/R}\omega)\ ,\ \omega \in \partial\Omega_R\ ,
\end{aligned}
$$

which coincides with (2.19).

Notice that for $t > 0$ the Lax semigroups have their values in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$. This explicitly follows from (2.12), i.e. from the fact that the spectrum of the semigroup generator $\sigma(\Lambda_{I,\partial\Omega_R}) := \{\lambda_{l,m} = \ l/R\}_{l=0,|m|\leq l}^{\infty}$ is discrete and

$$
\operatorname{Tr} S(t) = \sum_{l=0}^{\infty}(2l+1)\; e^{-l/R} < \infty\ . \qquad (2.25)
$$

The last is proven in the whole generality in Theorem 2.14.

3. Product approximations of Dirichlet-to-Neumann semigroups

3.1 Approximating family

Since in contrast to the Lax semigroup ($\gamma = I$) the action of the general Dirichlet-to-Neumann semigroup for $\gamma \neq I$ is known only implicitly (2.13), it is useful to construct converging approximations, which are simpler for calculations and analysis.

One of them is the Emamirad-Laadnani *approximation* [EmLa], which is motivated by the explicit action (2.19), (2.24) of the Lax semigroup

$$
\begin{aligned}
&(S(t)f)(\omega) = (T_{\partial\Omega_R} K(t) L_{\partial\Omega_R} f)(\omega) = v_f(e^{-t/R}\omega),\ \omega \in \partial\Omega_R\ , \qquad (3.1)\\
&K_R(t) : v(x) \mapsto v(e^{-t/R}\ x) \ \text{ for any } \ v \in C(\Omega_R)\,(\text{or}\,\mathcal{H}(\Omega_R))\ .
\end{aligned}
$$

The suggestion of [EmLa] consists in substitution of the family $\{K(t)\}_{t\geq 0}$ by the $\gamma-$*deformed* operator family:

$$
K_{\gamma,R}(t) : v(x) \mapsto v(e^{-(t/R)\ \gamma(x)}\ x) \ \text{ for any } \ v \in C(\Omega_R)\,(\text{or}\,\mathcal{H}(\Omega_R))\ . \qquad (3.2)
$$

Definition 3.1. *For the ball Ω_R the Emamirad-Laadnani approximating family $\{V_{\gamma,R}(t) := V_{\gamma,\partial\Omega_R}(t)\}_{t\geq 0}$ is defined by*

$$
(V_{\gamma,R}(t)f)(\omega) := (T_{\partial\Omega_R} K_\gamma(t) L_{\partial\Omega_R} f)(\omega) = v_f(e^{-(t/R)\ \gamma(\omega)}\ \omega),\ \omega \in \partial\Omega_R\ . \qquad (3.3)
$$

Remark 3.2. (a) *Notice that the approximating family (3.3) is not a semigroup:*

$$
\begin{aligned}
&(V_{\gamma,R}(t)V_{\gamma,R}(s)f)(\omega) = (T_{\partial\Omega_R} K_\gamma(t) L_{\partial\Omega_R} \widetilde{f}(s))(\omega) = \qquad (3.4)\\
&v_{\widetilde{f}(s)}(e^{-(t/R)\ \gamma(\omega)}\ \omega) \neq v_f(e^{-((t+s)/R)\ \gamma(\omega)}\ \omega) = (V_{\gamma,R}(t+s))f)(\omega)\ .
\end{aligned}
$$

(b) *This family is strongly continuous at* $t = 0$*:*

$$\lim_{t \searrow 0} V_{\gamma,R}(t)f = f \quad \text{for any} \quad f \in \partial X \; (\text{or}\, \partial\mathcal{H}) \; . \tag{3.5}$$

(c) *By definition (3.3) this family has derivative at* $t = +0$*:*

$$(\partial_t V_{\gamma,R}(t)f)(\omega) \mid_{t=0} = -\nu(\omega) \cdot \gamma(\omega)(\nabla v_f)(\omega) = -(\Lambda_{\gamma,\partial\Omega_R} f)(\omega) \; , \tag{3.6}$$

which for any $f \in \mathrm{dom}(\Lambda_{\gamma,\partial\Omega_R})$ *coincides with the (minus) Dirichlet-to-Neumann operator (2.3).*

3.2 Strong approximation of the Dirichlet-to-Neumann semigroups

By virtue of Remark 3.2 the Emamirad-Laadnani approximation family verifies the conditions of the Chernoff *approximation theorem* (Theorem 1.1, [Che]):

Proposition 3.3. *Let* $\{\Phi(s)\}_{s\geq 0}$ *be a family of linear contractions on a Banach space* $\mathfrak{B}$ *and let* X_0 *be the generator of a* C_0*-contraction semigroup. Define* $X(s) := s^{-1}(I - \Phi(s))$, $s > 0$. *Then for* $s \to +0$ *the family* $\{X(s)\}_{s>0}$ *converges strongly in the resolvent sense to the operator* X_0 *if and only if the sequence* $\{\Phi(t/n)^n\}_{n\geq 1}$, $t > 0$, *converges strongly to* e^{-tX_0} *as* $n \to \infty$, *uniformly on any compact* t*-intervals in* $\mathbb{R}^1_+$.

Notice that $\{V_{\gamma,R}(t)\}_{t\geq 0}$ in the Banach space ∂X is the family of contractions because of the *maximum principle* for the γ-harmonic functions v_f. Since the Dirichlet-to-Neumann operator (2.3) is densely defined and closed, Remark 3.2 (c) implies that the family $X(s) := s^{-1}(I - V_{\gamma,R}(s))$ converges for $s \to +0$ to $X_0 = \Lambda_{\gamma,\partial\Omega_R}$ in the strong resolvent sense.

The similar arguments are valid for the case of the Hilbert space $\partial\mathcal{H}$. By virtue of Remark 2.4 the Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$ is non-negative and self-adjoint. This implies again that (3.3) is the family of contractions in $\partial\mathcal{H}$ and that by Remark 3.2 (c) the family $X(s) := s^{-1}(I - V_{\gamma,R}(s))$ converges for $s \to +0$ to $X_0 = \Lambda_{\gamma,\partial\Omega_R}$ in the *strong* resolvent sense.

Resuming the above observations we obtain the *strong* approximation of the Dirichlet-to-Neumann semigroup $U(t)$:

Corollary 3.4. [EmLa]

$$\lim_{n\to\infty} (V_{\gamma,R}(t/n))^n f = U(t)f \; , \quad \textit{for every } f \in \partial X \textit{ or } \partial\mathcal{H} \; , \tag{3.7}$$

uniformly on any compact t*-intervals in* $(0, \infty)$.

The Emamirad-Laadnani approximation theorem (Corollary 3.4) has the following important extension to more *general* geometry than ball [EmLa].

Definition 3.5. *We say that a bounded smooth domain* Ω *in* $\mathbb{R}^d$ *has the* property of the interior ball, *if for any* $\omega \in \partial\Omega$ *there exists a tangent to* $\partial\Omega$ *at* ω *plane* $\mathcal{T}_\omega$, *and such that one can construct a ball tangent to* $\mathcal{T}_\omega$ *at* ω, *which is totally included in* Ω.

If Ω has this property, then with any point $\omega \in \partial\Omega$, one can associate a *unique* point x_ω, which is the center of the *biggest* ball $B(x_\omega, r_\omega)$ of radius r_ω included in Ω. For any $0 < r \leq r_\omega$, we can construct the approximating family $V_r(t)$ related to the ball $B(x_{r,\omega}, r) := \{x \in \Omega : |x - x_{r,\omega}| \leq r\}$ of radius r, which is centered on the line perpendicular to $\mathcal{T}_\omega$ at the point $\omega \in \partial\Omega$, i.e. $x_{r,\omega} = (r/r_\omega)x_\omega + (1 - r/r_\omega)\omega$. Then we define

$$(V_{\gamma,r}(t)f)(\omega) := T_{\partial\Omega}\ v_f\left(x_{r,\omega} + \mathrm{e}^{-(t/r)\gamma(\omega)}(r\ \nu_\omega)\right)\ . \tag{3.8}$$

Here ν_ω is the outer-normal vector at ω, the function $v_f = L_{\partial\Omega} f$ is the γ-harmonic lifting of the boundary condition f on $\partial\Omega$, and $T_{\partial\Omega}$ is the trace operator:

$$T_{\partial\Omega} : H^1(\Omega) \ni v \longmapsto v\ |_{\partial\Omega} \in H^{1/2}(\partial\Omega). \tag{3.9}$$

Remark 3.6. *Notice that:*
(a) *since* $\nu_\omega = (\omega - x_{r,\omega})/r$, *one gets* $(V_{\gamma,r}(t=0)f)(\omega) := (T_{\partial\Omega}\ v_f)(\omega) = f(\omega)$ *;*
(b) *by virtue of (3.8) the strong derivative at* $t = 0$ *has the form:*

$$(\partial_t V_{\gamma,r}(t=0)f)(\omega) = -\gamma(\omega)\nu_\omega \cdot (\nabla v_f)(\omega) = -(\Lambda_{\gamma,\partial\Omega} f)(\omega),$$

see (3.6).

Proposition 3.7. [EmLa] *Let* Ω *has the property of interior ball, and let*

$$\inf_{\omega\in\partial\Omega}\{r > 0 : B(x_\omega, r_\omega) \subset \Omega\} > 0,$$
$$\sup_{\omega\in\partial\Omega}\{r > 0 : B(x_\omega, r_\omega) \subset \Omega\} < \infty\ .$$

For any $0 < s \leq 1$ *we define* V_{γ,sr_ω}, *i.e.*

$$V_{\gamma,sr_\omega} f(\omega) = v_f\left(x_{s,\omega} + \mathrm{e}^{-(t/(sr_\omega))\gamma(\omega)}(sr_\omega\ \nu_\omega)\right)\ , \tag{3.10}$$

where $x_{s,\omega} = sx_\omega + (1-s)\omega$. *Then for any* $0 < s \leq 1$

$$\lim_{n\to\infty}(V_{\gamma,sr_\omega}(t/n))^n f = U(t)f\ , \quad \textit{for every } f \in \partial X \textit{ or } \partial\mathcal{H}\ , \tag{3.11}$$

uniformly on any compact t*-intervals in* $(0,\infty)$.

Remark 3.8. *By Definition 3.1 for the ball* Ω_R *and constant matrix-valued function* $\gamma(x) = I$ *one obviously have* $V_{\gamma=I,R}(t) = S(t) = U(t)$. *On the other hand, for a general smooth domain* Ω *with geometry verifying the conditions of Proposition 3.7, one is obliged to consider the family of approximations* V_{γ,sr_ω} *even for the homogeneous case* $\gamma = I$.

4. Dirichlet-to-Neumann Gibbs semigroups

4.1 Gibbs semigroups

Since by Lemma 2.14 for any Dirichlet-to-Neumann semigroup we obtain: $U(t > 0) \in \mathfrak{C}_1(\partial\mathcal{H})$, then one can check that it is in fact a *Gibbs* semigroup. To this end we recall main definitions and some results that we need for the proof, see e.g. [Zag2].

Let $\mathfrak{H}$ be a separable, infinite-dimensional complex Hilbert space. We denote by $\mathcal{L}(\mathfrak{H})$ the algebra of all bounded operators on $\mathfrak{H}$ and by $\mathfrak{C}_\infty(\mathfrak{H}) \subset \mathcal{L}(\mathfrak{H})$ the subspace of all *compact* operators. The $\mathfrak{C}_\infty(\mathfrak{H})$ is a $*$-*ideal* in $\mathcal{L}(\mathfrak{H})$, that is: if $A \in \mathfrak{C}_\infty(\mathfrak{H})$, then $A^* \in \mathfrak{C}_\infty(\mathfrak{H})$ and, if $A \in \mathfrak{C}_\infty(\mathfrak{H})$ and $B \in \mathcal{L}(\mathfrak{H})$, then $AB \in \mathfrak{C}_\infty(\mathfrak{H})$ and $BA \in \mathfrak{C}_\infty(\mathfrak{H})$. We say that a compact operator $A \in \mathfrak{C}_\infty(\mathfrak{H})$ belongs to the *von Neumann-Schatten* $*$-ideal $\mathfrak{C}_p(\mathfrak{H})$ for a certain $1 \leq p < \infty$, if the norm

$$\|A\|_p := \left(\sum_{n\geq 1} s_n(A)^p\right)^{1/p} < \infty, \tag{4.1}$$

where $s_n(A) := \sqrt{\lambda_n(A^*A)}$ are the *singular* values of A, defined by the eigenvalues $\{\lambda_n(\cdot)\}_{n\geq 1}$ of non-negative self-adjoint operator A^*A. Since the norm $\|A\|_p$ is a non-increasing function of $p > 0$, one gets:

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q > \|A\|_\infty (= \|A\|), \tag{4.2}$$

for $1 \leq p \leq q < \infty$. Then for the von Neumann-Schatten ideals this implies inclusions:

$$\mathfrak{C}_1(\mathfrak{H}) \subseteq \mathfrak{C}_p(\mathfrak{H}) \subseteq \mathfrak{C}_q(\mathfrak{H}) \subset \mathfrak{C}_\infty(\mathfrak{H}). \tag{4.3}$$

Let $p^{-1} = q^{-1} + r^{-1}$. Then by virtue of the *Hölder inequality* applied to (4.1) one gets: $\|AB\|_p \leq \|A\|_q \|B\|_r$, if $A \in \mathfrak{C}_q(\mathfrak{H})$ and $B \in \mathfrak{C}_r(\mathfrak{H})$. Consequently we obtain:

Lemma 4.1. *The operator A belongs to the trace-class $\mathfrak{C}_1(\mathfrak{H})$ if and only if there exists two* (Hilbert-Schmidt) *operators $K_1,\ K_2 \in \mathfrak{C}_2(\mathfrak{H})$, such that $A = K_1\ K_2$. Similarly, if $K \in \mathfrak{C}_p(\mathfrak{H})$, then $K^p \in \mathfrak{C}_1(\mathfrak{H})$.*

Let K be integral operator in the Hilbert space $L^2(D, \mu)$. It is a Hilbert-Schmidt operator if and only if its kernel $k(x, y) \in L^2(D \times D, \mu \times \mu)$ and then one gets the estimate: $\|K\|_2 \leq \|k\|_{L^2(D\times D,\mu\times\mu)}$.

The proof is quite straightforward and can be found in, e.g., [Kat], [Sim].

Definition 4.2. [Zag2] *Let $\{G(t)\}_{t\geq 0}$ be a C_0-semigroup on $\mathfrak{H}$ with $\{G(t)\}_{t>0} \subset \mathfrak{C}_\infty(\mathfrak{H})$. It is called immediate Gibbs semigroup, if $G(t) \in \mathfrak{C}_1(\mathfrak{H})$ for any $t > 0$ and it is called eventually Gibbs semigroup, if there is $t_0 > 0$, such that $G(t) \in \mathfrak{C}_1(\mathfrak{H})$ for any $t \geq t_0$.*

Remark 4.3. (a) *Notice that by Lemma 4.1 any C_0-semigroup such that one has $\{G(t)\}_{t>0} \subset \mathfrak{C}_p(\mathfrak{H})$ for some $p < \infty$, is an immediate Gibbs semigroup.*
(b) *Since compact C_0-semigroups are norm-continuous for any $t > 0$, the immediate Gibbs semigroups are $\|\cdot\|_1$-norm continuous for $t > 0$.*

For more details of the Gibbs semigroups properties we refer to the book [Zag2].

Corollary 4.4. *By virtue of Proposition 2.12, Definition 4.2 and Remark 4.3 the Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_t$ on the Hilbert space $\partial\mathcal{H}$ is a $\|\cdot\|_1$-holomorphic quasi-sectorial immediate Gibbs for* $\mathrm{Re}\,(t) > 0$.

4.2 Compact and Tr-norm approximating family

Proposition 4.5. [EmLa] *For the ball Ω_R the Emamirad-Laadnani approximating family $\{V_{\gamma,R}(t)\}_{t\geq 0}$ consists of compact operators on the Banach space $\partial X = C(\partial\Omega_R)$ for any $t > 0$.*

The proof follows from Definition 3.1 by *Arzela-Ascoli* criterium of compactness, since representation (3.3) and conditions on γ imply the uniform bound and equicontinuity of the sets $\{V_{\gamma,R}(t)(\partial X)\}_t$ for any $t > 0$.

For the case of the Hilbert space we recall the following useful condition for characterization of the Tr-class operators [Zag2].

Proposition 4.6. *If $A \in \mathcal{L}(\mathfrak{H})$ and $\sum_{j=1}^{\infty} \|Ae_j\| < \infty$ for an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of $\mathfrak{H}$, then $A \in \mathfrak{C}_1(\mathfrak{H})$.*

Theorem 4.7. *On the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R)$ the approximating family $\{V_{\gamma,R}(t)\}_{t>0} \subset \mathfrak{C}_1(\partial\mathcal{H})$.*

Proof: Since the eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ of the self-adjoint Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega_R}$ form an orthonormal basis in $L^2(\partial\Omega_R)$, we apply Proposition 4.6 for this basis.

Let $\partial\Omega_{t,\gamma,R} := \{x_\omega := e^{-(t/R)\ \gamma(\omega)}\ \omega\}_{\omega\in\partial\Omega_R}$. By representation (3.3) and by estimate (2.9) one obtains

$$\begin{aligned}\|V_{\gamma,R}(t)\phi_k\|^2 &= \int_{\partial\Omega_R} d\sigma(\omega)|v_{\phi_k}(x_\omega)|^2 \\ &\leq |\partial\Omega_R| \sup_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t,\gamma,R})^2/k^{2p/(d-1)}\ . \end{aligned} \tag{4.4}$$

Then by hypothesis **(H2)** on the matrix γ one gets for the norm of the vector x_ω in $\mathbb{R}^d$ the estimate:

$$\|x_\omega\| \leq \|e^{-(t/R)\ \gamma}\|\ R \leq e^{-c_1(t/R)}\, R\ .$$

Hence, for any $t > 0$ the $\mathrm{dist}(x_\omega, \partial\Omega_R) \geq (1 - e^{-c_1(t/R)})R > 0$, which implies for the estimates in (2.9) and in (4.4) that

$$0 < \inf_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t>0,\gamma,R}) \leq \sup_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t>0,\gamma,R}).$$

Then for $2p/(d-1) > 1$ the estimate (4.4) ensures the convergence of the series in the inequality:

$$\|V_{\gamma,R}(t)\|_1 \leq \sum_{k=1}^{\infty} \|V_{\gamma,R}(t)\phi_k\|,$$

which finishes the proof. □

5. Concluding remarks: trace-norm approximations

The strong Emamirad-Laadnani approximation theorem (Corollary 3.4) and the results of Section 4.2 proving that Dirichlet-to-Neumann semigroup $U(t)$ and approximants $V_{\gamma,\partial\Omega}(t/n)^n$ belong to $\mathfrak{C}_1(\partial\mathcal{H})$, for all $n \geq 1$ and $t > 0$, motivate the following conjecture:

Conjecture 5.1. [EmZa] *The Emamirad-Laadnani approximation theorem is valid in the* Tr*-norm topology of* $\mathfrak{C}_1(\partial\mathcal{H})$.

Remark 5.2. *Notice that the strong approximation of the Dirichlet-to-Neumann Gibbs semigroup* $U(t)$ *by the* Tr*-class family* $(V_{\gamma,\partial\Omega}(t/n))^n$ *does not lift automatically the topology of convergence to, e.g., operator-norm approximation* [Zag2].

Therefore, to prove the Conjecture 5.1 one needs additional arguments similar to those of [CaZag2]. To this end we put the difference in question $\Delta_n(t) := (V_{\gamma,\partial\Omega}(t/n))^n - U(t)$ in the following form:

$$\begin{aligned} \Delta_n(t) &= \{(V_{\gamma,\partial\Omega}(t/n))^{k_n} - (U(t/n))^{k_n}\}(V_{\gamma,R}(t/n))^{m_n} \\ &+ (U(t/n))^{k_n}\{(V_{\gamma,\partial\Omega}(t/n))^{m_n} - (U(t/n))^{m_n}\}. \end{aligned} \tag{5.1}$$

Here for any $n > 1$, we define two variables $k_n = [n/2]$ and $m_n = [(n+1)/2]$, where $[x]$ denotes the *integer part* of $x \geq 0$, i.e., $n = k_n + m_n$. Then for the estimate of $\Delta_n(t)$ in the $\mathfrak{C}_1(\partial\mathcal{H})-$topology one gets:

$$\begin{aligned} \|\Delta_n(t)\|_1 &\leq \|(V_{\gamma,\partial\Omega}(t/n))^{k_n} - (U(t/n))^{k_n}\| \, \|(V_{\gamma,\partial\Omega}(t/n))^{m_n}\|_1 \\ &+ \|(U(t/n))^{k_n}\|_1 \, \|(V_{\gamma,\partial\Omega}(t/n))^{m_n} - (U(t/n))^{m_n}\|. \end{aligned} \tag{5.2}$$

In spite of Remark 5.2, the *explicit* representation of approximants $\{(V_{\gamma,\partial\Omega}(t/n))^n\}_{n\geq 1}$ allows to prove the corresponding operator-norm estimate.

Theorem 5.3. [EmZa] *Let* $V_{\gamma,\partial\Omega_R}(t)$ *be defined by (3.3). Then one gets the estimate:*

$$\|(V_{\gamma,\partial\Omega_R}(t/n))^n - U(t)\| \leq \varepsilon(n) \ , \quad \lim_{n\to\infty} \varepsilon(n) = 0, \tag{5.3}$$

uniformly for any t*-compact in* $\mathbb{R}^1_+$.

To establish (5.3) we use the "telescopic" representation:

$$\begin{aligned} &(V_{\gamma,\partial\Omega_R}(t/n))^n - U(t) = \\ &\sum_{s=0}^{n-1} (V_{\gamma,\partial\Omega_R}(t/n))^{(n-s-1)}\{V_{\gamma,\partial\Omega_R}(t/n) - U(t/n)\}(U(t/n))^s \ , \end{aligned} \tag{5.4}$$

and the operator-norm estimate of $\{V_{\gamma,\partial\Omega_R}(t/n) - U(t/n)\}$ for large n.

The next auxiliary result establishes a relation between family of operators $V_{\gamma,\partial\Omega_R}(t)$ and the Dirichlet-to-Neumann semigroup $U(t)$.

Lemma 5.4. [EmZa] *There exists a bounded operator* $W_{\gamma,\partial\Omega_R}(t)$ *on* $\partial\mathcal{H}$ *such that*

$$V_{\gamma,\partial\Omega_R}(t) = W_{\gamma,\partial\Omega_R}(t)U(t) \ , \tag{5.5}$$

for any $t \geq 0$.

Now we return to the main inequality (5.2). To estimate the *first* term in the right-hand side of (5.2) we need Theorem 5.3 and the *Ginibre-Gruber* inequality [CaZag2]:

$$\|(V_{\gamma,\partial\Omega}(t/n))^{m_n}\|_1 \leq C\ U(m_n t/n)\ .$$

To establish the latter we use representation (5.5) given by Lemma 5.4.

To estimate the *second* term one needs only the result of Theorem 5.3. All together this gives a proof of Conjecture 5.1 at least for the ball Ω_R.

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